

**Diophantine equation on sixth degree -
-type (6-4-4) with four terms on both sides**

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Abstract

There are numerical solutions available on Wolfram world of mathematics website (ref. # 1) for the equation, $(a^6+b^6+c^6+d^6)=(e^6+f^6+g^6+h^6)$. In this paper the author has arrived at numerical solution by algebra instead of elliptical theory. There are methods for the (6-4-4) equation in (ref #7) providing numerical solutions but parametric solutions are not shown. Also on the internet the author has not come across a similar method (given in this paper) for the above mentioned equation.

Consider the below equation:

$$(a^6 + b^6 + c^6 + d^6) = (e^6 + f^6 + g^6 + h^6) \quad \text{--- (1)}$$

We have the Identity:

$$u^6 + v^6 = (x^6 - 3uvx^2(2x^2 - 3ab) - 2(uv)^3) \quad \text{----- (2)}$$

Where, $x = (u+v)$

In equation (1) we take:

$$[m = a + b = e + f, n = c + d = g + h, ab = u, cd = v, ef = p, gh = q] \quad \text{---- (3)}$$

equation (1) after transposing we have:

$$(a^6 + b^6) - (e^6 + f^6) = (g^6 + h^6) - (c^6 + d^6) \quad \text{----- (4)}$$

substituting from (3) into (4) we get:

$$a^6 + b^6 = m^6 - 3um^2(2m^2 - 3u) - 2(u)^3$$

$$e^6 + f^6 = m^6 - 3vm^2(2m^2 - 3p) - 2(p)^3$$

$$g^6 + h^6 = n^6 - 3qn^2(2n^2 - 3q) - 2(q)^3$$

$$c^6 + d^6 = n^6 - 3vn^2(2n^2 - 3v) - 2(v)^3$$

substituting above four equations in eqn (4) & simplifying we get:

$$(p - u)(6m^4 - 9m^2(p + u) + 2(p^2 + pu + u^2)) = (v - q)(6n^4 - 9n^2(v + q) + 2(v^2 + vq + q^2)) \text{ ----- (5)}$$

In-order to solve eqn. (5) we apply the condition: $(p - u) = (v - q)$ or

$$(p + q) = (u + v) \text{ ----- (6)}$$

hence we get from eqn. (5):

$$6m^4 - 9m^2(p + u) + 2(p^2 + pu + u^2) = 6n^4 - 9n^2(v + q) + 2(v^2 + vq + q^2) \text{ --- (7)}$$

The author noticed that the terms, $[9m^2(p + u)]$ & $[9n^2(v + q)]$ can be eliminated by taking:

$$2n^2 = -(p + u) \quad \& \quad 2m^2 = -(v + q) \text{ ----- (8)}$$

hence we get:

$$6m^4 + 2(p^2 + pu + u^2) = 6n^4 + 2(v^2 + vq + q^2) \text{ or,}$$

$$3m^4 + (p^2 + pu + u^2) = 3n^4 + (v^2 + vq + q^2) \text{ ----- (9)}$$

since we have: $(p-u)=(v-q)$ we get:

$$(v^2 + vq + q^2) - (p^2 + pu + u^2) = 3(vq - pu)$$

substituting above in (9) we get:

$$(m^4 - n^4) = (vq - pu) \text{ ----- (10)}$$

hence we have the three above conditions to satisfy:

namely equations, (6), (8) & (10)

$$\text{From eqn (8) we have: } -(p + u) = 2n^2 \quad \& \quad -(v + q) = 2m^2$$

$$\text{hence, } 2(m^2 + n^2) = -[(p + u) + (v + q)] = -[(p + q) + (u + v)]$$

since, from eqn (6) we have: $(p + q) = (u + v)$, we get:

$$2(m^2 + n^2) = [(p + u) + (v + q)] = [(p + q) + (u + v)] = -2(u + v)$$

hence, $(m^2 + n^2) = -(u + v)$ ----- (11)

we square both sides of eqn. (11) & we get:

$$m^4 = (u + v)^2 - 2m^2n^2 - n^4 \text{ ---- (12)}$$

we also have from eqn (10):

$$(m^4 - n^4) = (vq - pu)$$

substituting for (m^4) , from eqn (12) in above we get a quadratic in the variable "u" :

$$u^2 + u(2v + p) + (v^2 - vq - 2n^2(m^2 + n^2)) = 0 \text{ ----- (13)}$$

In-order to get integer solutions to eqn (1), the determinant in equation (13),

needs to be a square:

hence the determinant, (w^2) , is given by:

$$w^2 = 4v(p + q) + p^2 + 8n^2(m^2 + n^2) \text{ - - - - - (14)}$$

for eqn (14), the integer solution is:

$$(m, n, v, p, q) = (167, 148, -42485, -36500, -13293)$$

substituting above in eqn (13) we get: $u = -7308$

$$\text{hence, } (m, n, u, v, p, q) = (167, 148, -7308, -42485, -36500, -13293) \text{ - - - - - (15)}$$

we also have the equality:

$$\begin{aligned} m &= (a + b) = (e + f) \quad \&, \\ n &= (c + d) = (g + h) \text{ ----- (16)} \end{aligned}$$

$$\& \quad (m, n) = (167, 148)$$

Also we have:

$$u = ab, \quad v = cd, \quad p = ef, \quad q = gh \text{ ----- (17)}$$

hence from (15),(16) & (17) we get:

$$m = a + b = 167, \quad u = (a * b) = -(7308)$$

$$\text{solving above for (a,b), we get, } (a, b) = (203, -36) \text{ ----- (18)}$$

Next:

$$n = c + d = 148, \quad v = (c * d) = -(42485)$$

solving above for (c, d) , we get: $(c, d) = (293, -145)$ ----- (19)

Next:

$$m = e + f = 167, p = (e * f) = -(36500)$$

solving above for (e, f) , we get: $(e, f) = (292, -125)$ ----- (20)

Next:

$$n = g + h = 148, q = (g * h) = -(13293)$$

solving above for (g, h) , we get: $(g, h) = (211, -63)$ ----- (21)

hence from, eqn. (18),(19),(20),(21) we get:

$$(a, b, c, d) = (203, -36, 293, -145) \quad \&$$

$$(e, f, g, h) = (292, -125, 211, -63)$$

Therefore:

$$(a, b, c, d)^6 = (e, f, g, h)^6 \quad \text{or,}$$

$$(203, -36, 293, -145)^6 = (292, -125, 211, -63)^6 \quad \text{----- (22)}$$

above equation (22), can also be written as:

$$(203, -36, 293, -145)^k = (292, -125, 211, -63)^k \text{ ----- (23)}$$

Note:

The author noticed that equation (23) is also valid for the degree's, $(k=1,2,4,6)$.

Conclusion:

Since the degree six, is an even power, the negative signs in the numerical solution of equation (22) can be removed & the equation can be written as below:

$$(203, 36, 293, 145)^6 = (292, 125, 211, 63)^6$$

Also others can attempt to see if a parametric solution is possible for equation (1), which is shown above.

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